Examples of algebraic equations

1. Solve the equation
$$z^2 - 4z + 5 = 0.$$
We calculate $\Delta = -4$, so $\sqrt{\Delta} = \sqrt{4i^2} = \pm 2i$, and $z_1 = \frac{4+2i}{2} = 2 + i$, $z_2 = \frac{4-2i}{2} = 2 - i$. Because this was an equation with real coefficients and $\Delta < 0$, its roots are conjugate numbers.

2. Solve the equation
$$z^2 + (-1 + i)z + (2 + i) = 0.$$
Like above: $\Delta = -8 - 6i$. To calculate $\sqrt{\Delta}$ we use the formulas
$$\sqrt{a+bi} = \begin{cases} \pm\sqrt{a} & \text{for} \quad b = 0, a \geqslant 0, \\ \pm\sqrt{-a}i & \text{for} \quad b = 0, a < 0, \\ \pm\left(\sqrt{\frac{a+|z|}{2}} + i\operatorname{sgn} b\sqrt{\frac{-a+|z|}{2}}\right) & \text{for} \quad b \neq 0. \end{cases}$$
so $\sqrt{\Delta} = \pm(1 - 3i)$. Finally
$$z_1 = \frac{1 - i + 1 - 3i}{2} = 1 - 2i, \qquad z_2 = \frac{1 - i - 1 - 3i}{2} = i.$$
Now the roots are not conjugate numbers.

2. Solve the equation
$$z^3 - 4z^2 + 14z - 20 = 0.$$
This is an equation with integer coefficients. First we look for roots among divisors of 20. We find (good way is to apply Horner's scheme) $z_1 = 2$ and we can factorize
$$z^3 - 4z^2 + 14z - 20 = (z - 2)(z^2 - 2z + 10).$$
Now we solve $z^2 - 2z + 10 = 0$ using usual method.

3. Solve the equation
$$z^4 - 2z^2 + 4 = 0$$
Let $t = z^2$. Then
$$t^2 - 2t + 4 = 0.$$
Using usual method we get
$$t_1 = 1 + \sqrt{3}i, \quad t_2 = 1 - \sqrt{3}i.$$
For each of these numbers we have to find its square roots (best way is to use formulas above). The final answer is
$$z_1 = \frac{1}{\sqrt{2}}(\sqrt{3} + i),\ z_2 = -\frac{1}{\sqrt{2}}(\sqrt{3} + i),\ z_3 = \frac{1}{\sqrt{2}}(\sqrt{3} - i),\ z_4 = \frac{1}{\sqrt{2}}(-\sqrt{3} + i).$$

4. Solve the equation. Then plot its roots on the complex plane.
$$z^3 - (1 + i)^3 = 0,$$
Method 1.

We use the formula $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, which gives:

$$(z-1-i)(z^2+(1+i)z+(1+i)^2) = (z-1-i)(z^2+(1+i)z+2i)$$

Hence $z_1 = 1+i$ or $z^2+(1+i)z+2i = 0$.
To solve the last equation we calculate $\Delta = -6i$. To calculate $\sqrt{\Delta}$ we use the the above formulas and find $\sqrt{\Delta} = \sqrt{3} - i\sqrt{3}$.
Finally $z_2 = \frac{-1-\sqrt{3}}{2} + i\frac{-1+\sqrt{3}}{2}$, $z_3 = \frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}$.
These points are vertices of equilateral triangle inscribed into the circle of radius $\sqrt{2}$. It is easy to plot the vertex z_1 and then the other vertices.
Method 2.
Dividing the original equation by $(1+i)^3$ we obtain:

$$\left(\frac{z}{1+i}\right)^3 = 1.$$

Substituting w for $\frac{z}{1+i}$ we get the equation

$$w^3 = 1.$$

Using the polar form of 1, it is easy to find its cube roots, which are

$$w_1 = 1, \quad w_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad w_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

But $z = (1+i)w$, so now we multiply these numbers by $1+i$ to get the answer.