Properties of Functions

1. Finding formulas for functions.
1. Consider a triangle ABC, where $AC = b$ and $BC = a$ are constant, but the angle x at the vertex C changes. Find the formula for the area of this triangle as a function of x. Sketch the graph.
2. Find the formula for the volume of a cylinder inscribed into the sphere of a radius R as a function of:
a) a radius r of the base of the cylinder;
b) an altitude h of the cylinder.

2. Find the domain, the image, and sketch the graph of a function:
a) $y = \sqrt{(x+2)^2} + \sqrt{(x-1)^2}$
b) $y = \sqrt{1-\cos^2 x} + \sin x$
c) $y = \sqrt{x^2 - 4}$
d) $y = \sqrt{\ln(x^2-4)}$
e) $y = \ln \cos x$.

3. Even functions and odd functions.
1. A function $f(x)$ is odd and its domain is $\mathbb{R}$. What can you tell about a function (is it even or odd?):
a) $kf(x)$; b) $f(kx)$; k=const?
2. Complete the definition

$$f(x) = \begin{cases} 2x+1 & \text{for} \quad x > 0 \\ & \text{for} \quad x = 0 \\ & \text{for} \quad x < 0 \end{cases}$$

so as to get: a) an even function; b) an odd function. Sketch the graphs for both cases.
Remember that the graph of an even function is symmetric around Oy-axis, and the graph of an odd function is circularly symmetric, around the origin O.

4. Periodic functions.
A function f is said to be periodic with period T (T being a nonzero constant) if we have $f(x+T) = f(x)$ for all values of x in the domain. The most important examples are the trigonometric functions.
The graph of a periodic function is invariant under translation in the x-direction by a distance of T.
Periodic functions are used in physics to describe oscillations, waves, and other phenomena that exhibit periodicity.
1) Sketch the graph of a periodic function, which has the same graph as the function $y = x^2$:
a) on the interval $[0, 1)$; b) on the interval $[-\frac{1}{2}, \frac{1}{2}]$.
2) We define *Dirichlet function* as:

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Is this function periodic?
3) Show that if f is periodic with period T than g defined by $g(x) = f(3x)$ has a period $\frac{1}{3}T$.

5. Graphs of functions.
1. Find the graphs of functions:
a) $y = \cos(x + \frac{\pi}{4}) - 1$
b) $y = \log_3(3 - x)$
c) $y = 2^{x+3}$
d) $y = 2 \sin \frac{x}{2}$
e) $y = \frac{x}{x-2}$

6. Composite functions.
In mathematics, function composition is the pointwise application of one function to the result of another to produce a third function. For instance, the functions $f : X \to Y$ and $g : Y \to Z$ can be composed to yield a function which maps $x \in X$ to $g(f(x))$ in Z. Intuitively, if z is a function of y, and y is a function of x, then z is a function of x. The resulting composite function is denoted $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. Intuitively, composing two functions is a chaining process in which the output of the inner function becomes the input of the outer function.
The notation $g \circ f$ is read as "g circle f ", or "g round f ", or "g composed with f ", "g after f ", "g following f ", or "g of f", or "g on f ".
1. Find the compositions $f \circ f, f \circ g, g \circ f, g \circ g$, if:
a) $f(x) = \frac{1}{x}, g(x) = x^2$;
b) $f(x) = \log_2 x, g(x) = 2^x$.
c) $f(x) = \frac{1}{x-1}, g(x) = x^2 + 2x$;
b) $f(x) = x^3, g(x) = 2^x$.

7. Inverse function.
An inverse function is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa. i.e., $f(x) = y$ if and only if $g(y) = x$.
1. Show that a following function is one-to-one and find its inverse function:
a) $y = 3x + 4$;
b) $y = 10^x$;
c) $y = \log \log x$;
d) $y = x + 2\sqrt{x} + 1$.
2. For a following function find an interval on which it is one-to-one. Then find its inverse function:
a) $y = \frac{1}{x^2}$;
b) $y = \frac{1}{1+x^2}$;
c) $y = x^2 - 4x + 3$.
d) $y = x^2 - 2x + 3$
e) $y = 3^x + 3^{-x}$

8. Inverse trigonometric functions.
1. Find
a) $\arccos(-\frac{\sqrt{3}}{2}) + \arcsin 1 + \arctan \sqrt{3}$.
b) $2 \arccos(-\frac{1}{2}) + \arctan(\tan \frac{7}{8}\pi) - \arctan 1$.
2. Check the following identities
a) $\arcsin x + \arccos x = \frac{\pi}{2}$ dla $x \in [-1, 1]$
b) $\cos(\arcsin x) = \sqrt{1 - x^2}$ dla $x \in [-1, 1]$
3. Find the domain, the image, and sketch the graph of a function:
a) $y = \arcsin(x - 2)$
b) $y = 3 + \arctan(x + 2)$
c) $y = 1 + \frac{1}{2} \arccos(2x - 1)$

9. Hyperbolic functions.
Let $e \approx 2.718\ldots$ be the Euler's number. We define **hyperbolic sine** *as*

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

and **hyperbolic cosine** as

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

1. Prove that
a) $\cosh^2 x - \sinh^2 x = 1$.
b) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
c) $\sinh(2x) = 2 \sinh x \cosh x$.