Integrals

Basic Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \in \mathbb{R}, n \neq -1 \tag{1}$$

$$\int \frac{1}{x} dx = \ln|x| + C \tag{2}$$

$$\int \sin x dx = -\cos x + C \tag{3}$$

$$\int \cos x dx = \sin x + C \tag{4}$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C \tag{5}$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C \tag{6}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \tag{7}$$

$$\int e^x dx = e^x + C \tag{8}$$

$$\int \frac{dx}{1+x^2} = \arctan x + C \tag{9}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \tag{10}$$

$$\int \sinh x dx = \cosh x + C \tag{11}$$

$$\int \cosh x dx = \sinh x + C \tag{12}$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x + C \tag{13}$$

$$\int \frac{dx}{\sinh^2 x} = -\coth x + C \tag{14}$$

and

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

Theorem 1 (Integration by Substitution) *If a function $f(t)$ is differentiable on an interval (a, b) and a function $t = g(x)$ has a continuous derivative on (α, β) and $a < g(x) < b$ for $x \in (\alpha, \beta)$, then*

$$\int f(g(x))\, g'(x) dx = \int f(t) dt.$$

Corollary 1 *If the function $F(x)$ is an antiderivative of a function $f(x)$, then*

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C$$

Theorem 2 (Integration by Parts) *If functions $u(x)$ and $v(x)$ have continuous derivatives on some interval I, then for $x \in I$*

$$\int u(x) v'(x) dx = u(x)v(x) - \int v(x) u'(x) dx.$$

Recall that $v'(x)dx = dv$, $u'(x)dx = du$ (differentials). Hence the formula may be written in the shorter form

$$\int u\,dv = uv - \int v\,du.$$

Integration by Reduction Formulas
Sometimes we will have an expression containing an integer parameter, usually in the form of powers of elementary functions, which is impossible to integrate directly. Then we can try to find a reduction formula (by using other methods of integration), in the form of a recurrence relation. Applying the found reduction formula we obtain the integral of the same or similar expression with a lower integer parameter, progressively simplifying the integral until it can be evaluated.
Here some useful reduction formulas.

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx, \quad n \geq 2, \tag{15}$$

$$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx, \quad n \geq 2, \tag{16}$$

$$\int x^n a^x dx = \frac{x^n a^x}{\ln a} - \frac{n}{\ln a}\int x^{n-1} a^x dx, \quad n \geq 1, \tag{17}$$

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)^{n-1}}, \quad n \geq 2, \tag{18}$$

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)a^2(a^2+x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(a^2+x^2)^{n-1}}, \quad n \geq 2, \tag{19}$$

Additional Formulas for Integrals

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \tag{20}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \tag{21}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \tag{22}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \tag{23}$$

$$\int \sqrt{a^2-x^2} = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{a^2-x^2} + C \tag{24}$$

$$\int \frac{dx}{\sqrt{x^2+a}} = \ln|x+\sqrt{x^2+a}| + C \tag{25}$$

$$\int \sqrt{x^2+a}dx = \frac{a}{2}\ln|x+\sqrt{x^2+a}| + \frac{x}{2}\sqrt{x^2+a} + C \tag{26}$$

Integration of Rational Functions by Partial Fractions
It is easy to find

$$\int \Big(\frac{2}{x-2} + \frac{3}{x+1}\Big)dx = 2\ln|x-2| + 3\ln|x+1| + C.$$

But the integral

$$\int \frac{5x+4}{x^2-x-2}dx,$$

does not seem so easy, until we note that

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}.$$

Consider integrating a fraction $\frac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ are polynomials.
By the last example, it looks like if we can somehow break the fraction down by reversing the process of finding a common denominator, we should be able to integrate certain rational functions. The method of reversing finding a common denominator is a very important tool in Calculus, so it has its own name - **The Method of Partial Fractions**.
Suppose that $\frac{P(x)}{Q(x)}$ is a rational functions where $P(x)$, $Q(x)$ are polynomials. The method of partial fractions is fairly difficult, so we start by describing the general set up and then proceed through the different cases. We always start by doing the following:
1. If the degree of $P(x)$ is greater or equal to the degree of $Q(x)$ use polynomial division to reduce $\frac{P(x)}{Q(x)}$ to an expression of the form

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $R(x)$ is a polynomial of the degree less than the degree of $Q(x)$
2. Factor $Q(x)$ completely. It is a fact from algebra that any polynomial can be factored to powers of linear functions (things of the form $(x-a)^k$) and powers of quadratics (things of the form $(ax^2+bx+c)^k$).
3. Express $\frac{R(x)}{Q(x)}$ as a sum of simple fractions. This is possible by the following theorem.

Theorem 3 (Partial-fraction decomposition) *Let* $\frac{R(x)}{Q(x)}$ *be a rational function with* $\deg R < \deg Q$. *If*

$$Q(x) = a(x-x_1)^{k_1}(x-x_2)^{k_2}\dots(x-x_r)^{k_r}(x^2+p_1x+q_1)^{l_1}(x^2+p_2x+q_2)^{l_2}\dots(x^2+p_sx+q_s)^{l_s},$$

then to the factor $(x-x_i)^{k_i}$ *it corresponds a sum of* k_i *simple fractions of a form*

$$\frac{A_1}{x-x_i} + \frac{A_2}{(x-x_i)^2} + \cdots \frac{A_{k_i}}{(x-x_i)^{k_i}},$$

and to the factor $(x^2+p_jx+q_j)^{l_j}$ *it corresponds a sum of* l_j *simple fractions of a form*

$$\frac{B_1x+C_1}{x^2+p_jx+q_j} + \frac{B_2x+C_2}{(x^2+p_jx+q_j)^2} + \cdots + \frac{B_{l_j}x+C_{l_j}}{(x^2+p_jx+q_j)^{l_j}}.$$

4. Write RHS on a common denominator and then write the equality

$$R(x) = \text{numerator of } RHS.$$

5. Compute values of constants using one of the methods:
Method 1: Eliminate all constants except one by choosing suitable values for x;
Method 2: Create a linear system for constants by comparing coefficients of powers on both sides of the equality.
After all this you need only to integrate simple fractions. However, this is not always easy! But always possible.