Exercises: logic and induction

1. For every statement below write their negation (with the symbol $\sim$), then transform them to form without $\sim$:
a) $\forall x \exists a \; x > a$;
b) $x < a \wedge x > b$;
c) $\forall x \exists a \exists b \; (x < a \wedge x > b)$;
d) $x > b \Rightarrow x > a$.
e) $\exists x \; x^3 - x = 0$;
f) $\forall x, y \; (y > x \vee y \leqslant x)$;
g) $\exists y \forall x \; x + y = 0$;
h) $\forall x \exists y \; x + y = 0$;
i) $\forall x \exists y \; (x - y)^2 = x^2 - y^2$.

2. The following formula says that *disjunction is distributive over conjunction*. Writing a truth table show that it is a tautology (that means it is true for all possible valuations):

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r).$$

3. Prove that the following formula, called *Indirect Reasoning* is a tautology.

$$[(p \Rightarrow q) \wedge (\sim q)] \Rightarrow \sim p).$$

4. Show that a common fallacy:

$$[(p \Rightarrow q) \wedge (\sim p)] \Rightarrow \sim q).$$

is not a law of logic.

5. Show by induction that a set of n elements has 2^n subsets.
Remark: write clearly 1. base case; 2. induction step; 3. conclusion

6. Show by induction that for every $n \in \mathbb{N}$: $1 + 3 + \cdots + (2n - 1) = n^2$

7. Show by induction that for every $n \in \mathbb{N}$ the number $7^n - 1$ can be divided by 6.